

## 9 GENERATORS OF THE SKEIN SPACE OF THE 3-TORUS

ALESSIO CARREGA

ABSTRACT. We show that the skein vector space of the 3-torus is finitely generated. We show that it is generated by 9 elements: the empty set, some simple closed curves representing the non null elements of the first homology group with coefficients in  $\mathbb{Z}_2$ , and a link consisting of two parallel copies of one of the previous non empty knots.

### 1. INTRODUCTION

An alternative approach to representation theory for *quantum invariants* is provided by *skein theory*. The word “skein” and the notion were introduced by Conway in 1970 for his model of the *Alexander polynomial*. This idea became really useful after the work of Kauffman [K] which redefined the *Jones polynomial* in a very simple and combinatorial way passing through the *Kauffman bracket*. These combinatorial techniques allow us to reproduce all quantum invariants arising from the representations of  $U_q(\mathfrak{sl}_2)$  without any reference to representation theory. This also leads to many interesting and quite easy computations. This skein method was used by various authors [L1, L2, L3, L4, BHMV, KL] to re-interpret and extend some of the methods of representation theory.

The first notion in skein theory is the one of “*skein vector space*” (or *skein module*). These are vector spaces ( $R$ -modules) associated to oriented 3-manifolds, where the base field is equipped with a fixed invertible element  $A$ . These were introduced independently in 1988 by Turaev [Tu] and in 1991 by Hoste and Przytycki [HP1]. We can think of them as an attempt to get an algebraic topology for knots: they can be seen as homology spaces obtained using isotopy classes instead of homotopy or homology classes. In fact they are defined taking a vector space generated by sub-objects (*framed links*) and then quotienting them by some relations. In this framework, the following questions arise naturally and are still open in general:

#### Question 1.1.

- Are skein spaces (modules) computable?
- How powerful are them to distinguish 3-manifolds and links?
- Do the vector spaces (modules) reflect the topology/geometry of the 3-manifolds (*e.g.* surfaces, geometric decomposition)?
- Does this theory have a functorial aspect? Can it be extended to a functor from a category of cobordisms to the category of vector spaces (modules) and linear maps?

Skein spaces (modules) can also be seen as deformations of the ring of the  $SL_2(\mathbb{C})$ -character variety of the 3-manifold [B2]. Moreover they are useful to generalize the Kauffman bracket, hence the Jones polynomial, to manifolds other than  $S^3$ . Thanks to result of Hoste-Przytycki [HP4, P2] and (with different techniques) to Costantino [C], now we can define the Kauffman bracket also in the connected sum  $\#_g(S^1 \times S^2)$  of  $g \geq 0$  copies of  $S^1 \times S^2$ .

Until now there are only few 3-manifolds whose skein space (module) is known, see for instance [B1, HP2, HP3, HP4, Ma, Mr1, Mr2, MrD, P1, P2, P3, TL]. Another natural question is:

**Question 1.2.** Is the skein vector space of a closed oriented 3-manifold always finite generated?

In this paper we take as base field the set  $\mathbb{Q}(A)$  of all rational functions with rational coefficients and abstract variable  $A$ , and then we note that every result in this work holds also for the field  $\mathbb{C}$  of complex numbers with  $A \in \mathbb{C}$  a non null number such that  $A^n \neq 1$  for every  $n > 0$ .

**Theorem 1.3.** *The skein space  $K(T^3)$  of the 3-torus  $T^3 = S^1 \times S^1 \times S^1$  is finitely generated.*

A set of 9 generators is given by the empty set  $\emptyset$ , some simple closed curves representing the non null elements of the first homology group  $H_1(T^3; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^3$  with coefficients in  $\mathbb{Z}_2$ , and a skein element  $\alpha$  that is equal to the link consisting of two parallel copies of any previous non empty knots.

Our main tool is the algebraic work of Frohman and Gelca [FG]. The skein space (module) of a (thickened) surface has a natural structure of algebra obtained by overlap of framed links. In their work Frohman and Gelca gave a nice formula that describes the product in the skein space (algebra)  $K(T^2)$  of the 2-torus  $T^2 = S^1 \times S^1$ . A standard embedding of  $T^2$  in  $T^3$  makes this product commutative, hence we can get further relations from the formula of Frohman-Gelca.

**Acknowledgments.** The author is warmly grateful to Bruno Martelli for his constant support and encouragement.

## 2. THE RESULT

**2.1. Definition of skein module.** Let  $M$  be an oriented 3-manifold,  $R$  a commutative ring with unit and  $A \in R$  an invertible element of  $R$ . Let  $V$  be the abstract free  $R$ -module generated by all framed links in  $M$  (considered up to isotopies) including the empty set  $\emptyset$ .

**Definition 2.1.** The  $(R, A)$ -Kauffman bracket skein module of  $M$ , or the  $R$ -skein module, or simply the  $KBSM$ , is sometimes indicated with  $KM(M; R, A)$ ,

and is the quotient of  $V$  by all the possible *skein relations*:

$$\begin{aligned} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} &= A \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + A^{-1} \begin{array}{c} \diagup \diagup \\ \diagdown \diagdown \end{array} \\ L \sqcup \bigcirc &= (-A^2 - A^{-2})D \\ \bigcirc &= (-A^2 - A^{-2})\emptyset \end{aligned} .$$

These are local relations where the framed links in an equation differ just in the pictured 3-ball that is equipped with a positive trivialization. An element of  $KM(M; R, A)$  is called a *skein* or a *skein element*. If  $M$  is the oriented  $I$ -bundle over a surface  $S$  (this is  $M = S \times [-1, 1]$  if  $S$  is oriented) we simply write  $KM(S; R, A)$  and call it the *skein module* of  $S$ .

Let  $\mathbb{Q}(A)$  be field of all rational function with rational coefficients and abstract variable  $A$ . We set

$$K(M) := KM(M; \mathbb{Q}(A), A)$$

and we call it the *skein vector space*, or simply the *skein space*, of  $M$ .

**Remark 2.2.** It is easy to verify that if we modify the framing of a component of a framed link, the skein changes by the multiplication of an integer power of  $-A^3$ :

$$\begin{aligned} \overbrace{\bigcirc} &= -A^3 \text{---} \\ \underbrace{\bigcirc} &= -A^{-3} \text{---} . \end{aligned}$$

## 2.2. The skein algebra of the 2-torus.

**Definition 2.3.** Let  $S$  be a surface the skein module  $KM(S; R, A)$  has a natural structure of  $R$ -algebra that is given by the linear extension of the multiplication defined on framed links. Given two framed links  $L_1, L_2 \subset S \times [-1, 1]$ , the product  $L_1 \cdot L_2 \subset S \times [-1, 1]$  is obtained by putting  $L_1$  above  $L_2$ ,  $L_1 \cdot L_2 \cap S \times [0, 1] = L_1$  and  $L_1 \cdot L_2 \cap S \times [-1, 0] = L_2$ .

Look at the 2-torus  $T^2$  as the quotient of  $\mathbb{R}^2$  modulo the standard lattice of translations generated by  $(1, 0)$  and  $(0, 1)$ , hence for any non null pair  $(p, q)$  of integers we have the notion of  $(p, q)$ -*curve*: the simple closed curve in the 2-torus that is the quotient of the line passing through  $(0, 0)$  and  $(p, q)$ .

**Definition 2.4.** Let  $p$  and  $q$  be two co-prime integers, hence  $(p, q) \neq (0, 0)$ . We denote by  $(p, q)_T$  the  $(p, q)$ -curve in the 2-torus  $T^2$  equipped with the black-board framing. Given a framed knot  $\gamma$  in an oriented 3-manifold  $M$  and an integer  $n \geq 0$ , we denote by  $T_n(\gamma)$  the skein element defined by induction as follows:

$$\begin{aligned} T_0(\gamma) &:= 2 \cdot \emptyset \\ T_1(\gamma) &:= \gamma \\ T_{n+1}(\gamma) &:= \gamma \cdot T_n(\gamma) - T_{n-1}(\gamma) \end{aligned}$$

where  $\gamma \cdot T_n(\gamma)$  is the skein element obtained adding a copy of  $\gamma$  to all the framed links that compose the skein  $T_n(\gamma)$ . For  $p, q \in \mathbb{Z}$  such that  $(p, q) \neq (0, 0)$ , we denote by  $(p, q)_T$  the skein element

$$(p, q)_T := T_{\text{MCD}(p, q)} \left( \left( \frac{p}{\text{MCD}(p, q)}, \frac{q}{\text{MCD}(p, q)} \right)_T \right),$$

where  $\text{MCD}(p, q)$  is the maximum common divisor of  $p$  and  $q$ . Finally we set

$$(0, 0)_T := 2 \cdot \emptyset.$$

It is easy to show that the set of all the skein elements  $(p, q)_T$  with  $p, q \in \mathbb{Z}$  generates  $KM(T^2; R, A)$  as  $R$ -module.

This is not the standard way to color framed links in a skein module. The colorings  $JW_n(\gamma)$ ,  $n \geq 0$ , with the Jones-Wenzl projectors are defined in the same way of  $T_n(\gamma)$  but at the 0-level we have  $JW_0(\gamma) = \emptyset$ .

**Theorem 2.5** (Frohman-Gelca). *For any  $p, q, r, s \in \mathbb{Z}$  the following holds in the skein module  $KM(T^2; R, A)$  of the 2-torus  $T^2$ :*

$$(p, q)_T \cdot (r, s)_T = A \begin{vmatrix} p & q \\ r & s \end{vmatrix} (p+r, q+s)_T + A^{-1} \begin{vmatrix} p & q \\ r & s \end{vmatrix} (p-r, q-s)_T,$$

where  $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$  is the determinant  $ps - qr$ .

*Proof.* See [FG]. □

### 2.3. The abelianization.

**Definition 2.6.** Let  $B$  be a  $R$ -algebra for a commutative ring with unity  $R$ . We denote by  $C(B)$  the  $R$ -module defined as the following quotient:

$$C(B) := \frac{B}{[B, B]}$$

where  $[B, B]$  is the sub-module of  $B$  generated by all the elements of the form  $ab - ba$  for  $a, b \in B$ . We call  $C(B)$  the *abelianization* of  $B$ .

**Remark 2.7.** Usually in non-commutative algebra the *abelianization* is the  $R$ -algebra defined as the quotient of  $B$  modulo the sub-algebra (sub-module and ideal) generated by all the elements of the form  $ab - ba$ . In our definition the denominator is just a sub-module and we only get a  $R$ -module. We use the word “abelianization” anyway.

Now we work with  $C(K(T^2))$  and we still use  $(p, q)_T$  and  $(p, q)_T \cdot (r, s)_T$  to denote the class of  $(p, q)_T \in K(T^2)$  and  $(p, q)_T \cdot (r, s)_T \in K(T^2)$  in  $C(K(T^2))$ .

**Lemma 2.8.** *Let  $(p, q)$  be a pair of integers different from  $(0, 0)$ . Then in the abelianization  $C(K(T^2))$  of the skein algebra  $K(T^2)$  of the 2-torus  $T^2$*

we have

$$(p, q)_T = \begin{cases} (1, 0)_T & \text{if } p \in 2\mathbb{Z} + 1, \ q \in 2\mathbb{Z} \\ (0, 1)_T & \text{if } p \in 2\mathbb{Z}, \ q \in 2\mathbb{Z} + 1 \\ (1, 1)_T & \text{if } p, q \in 2\mathbb{Z} + 1 \\ (2, 0)_T & \text{if } p, q \in 2\mathbb{Z} \end{cases}.$$

Hence  $C(K(T^2))$  is generated as a  $\mathbb{Q}(A)$ -vector space by the empty set  $\emptyset$ , the framed knots  $(1, 0)_T$ ,  $(0, 1)_T$ ,  $(1, 1)_T$ , and a framed link consisting of two parallel copies of  $(1, 0)_T$ .

*Proof.* By Theorem 2.5 for every  $p, q \in \mathbb{Z}$  we have

$$\begin{aligned} A^{-q}(p+2, q)_T + A^q(p, q)_T &= (p+1, q)_T \cdot (1, 0)_T \\ &= (1, 0)_T \cdot (p+1, q)_T \\ &= A^q(p+2, q)_T + A^{-q}(-p, -q)_T. \end{aligned}$$

Since  $(p, q)_T = (-p, -q)_T$  we have  $(A^q - A^{-q})(p, q)_T = (A^q - A^{-q})(p+2, q)_T$ . Hence if  $q \neq 0$  we get  $(p, q)_T = (p+2, q)_T$  (here we use the fact that the base ring is a field and  $A^n \neq 1$  for every  $n > 0$ ). Thus

$$(p, q)_T = \begin{cases} (0, q)_T & \text{if } p \in 2\mathbb{Z}, \ q \neq 0 \\ (1, q)_T & \text{if } p \in 2\mathbb{Z} + 1, \ q \neq 0 \end{cases}.$$

Analogously by using  $(0, 1)_T$  instead of  $(1, 0)_T$  for  $p \neq 0$  we get

$$(p, q)_T = \begin{cases} (p, 0)_T & \text{if } q \in 2\mathbb{Z}, \ q \neq 0 \\ (p, 1)_T & \text{if } q \in 2\mathbb{Z} + 1, \ q \neq 0 \end{cases}.$$

Therefore if  $p, q \in 2\mathbb{Z} + 1$ ,  $(p, q)_T = (1, 1)_T$ . If  $p \neq 0$  we get

$$(p, 0)_T = (p, 2)_T = \begin{cases} (0, 2)_T & \text{if } p \in 2\mathbb{Z} \\ (1, 2)_T & \text{if } p \in 2\mathbb{Z} + 1 \end{cases} = \begin{cases} (0, 2)_T & \text{if } p \in 2\mathbb{Z} \\ (1, 0)_T & \text{if } p \in 2\mathbb{Z} + 1 \end{cases}.$$

In the same way for  $q \neq 0$  we get

$$(0, q)_T = (2, q)_T = \begin{cases} (2, 0)_T & \text{if } p \in 2\mathbb{Z} \\ (2, 1)_T & \text{if } p \in 2\mathbb{Z} + 1 \end{cases} = \begin{cases} (2, 0)_T & \text{if } p \in 2\mathbb{Z} \\ (0, 1)_T & \text{if } p \in 2\mathbb{Z} + 1 \end{cases}.$$

In particular we have

$$(2, 0)_T = (2, 2)_T = (2, -2)_T = (0, 2)_T = (p, q)_T \text{ for } (p, q) \neq (0, 0), \ p, q \in 2\mathbb{Z}.$$

□

**2.4. The  $(p, q, r)$ -type curves.** As for the 2-torus  $T^2$ , we look at the 3-torus  $T^3$  as the quotient of  $\mathbb{R}^3$  modulo the standard lattice of translations generated by  $(1, 0, 0)$ ,  $(0, 1, 0)$  and  $(0, 0, 1)$ .

**Definition 2.9.** Let  $(p, q, r)$  be a triple of co-prime integers, that means  $\text{MCD}(p, q, r) = 1$ , where  $\text{MCD}(p, q, r)$  is the maximum common divisor of  $p$ ,  $q$  and  $r$ , in particular we have  $(p, q, r) \neq (0, 0, 0)$ . The  $(p, q, r)$ -curve is the simple closed curve in the 3-torus that is the quotient (under the standard

lattice) of the line passing through  $(0,0,0)$  and  $(p,q,r)$ . We denote by  $[p,q,r]$  the  $(p,q,r)$ -curve equipped with the framing that is the collar of the curve in the quotient of any plane containing  $(0,0,0)$  and  $(p,q,r)$ . The framing does not depend on the choice of the plane.

**Definition 2.10.** An embedding  $e : T^2 \rightarrow T^3$  of the 2-torus in the 3-torus is *standard* if it is the quotient (under the standard lattice) of a plane in  $\mathbb{R}^3$  that is the image of the plane generated by  $(1,0,0)$  and  $(0,1,0)$  under a linear map defined by a matrix of  $SL_3(\mathbb{Z})$  (a  $3 \times 3$  matrix with integer entries and determinant 1).

**Remark 2.11.** There are infinitely many standard embeddings even up to isotopies. A standard embedding of  $T^2$  in  $T^3$  is the quotient under the standard lattice of the plane generated by two columns of a matrix of  $SL_3(\mathbb{Z})$ .

**Lemma 2.12.** *Let  $(p,q,r)$  be a triple of co-prime integers. Then the skein element  $[p,q,r] \in K(T^3)$  is equal to  $[x,y,z]$ , where  $x,y,z \in \{0,1\}$  and have respectively the same parity of  $p, q$  and  $r$ .*

*Proof.* Every embedding  $e : T^2 \rightarrow T^3$  of the 2-torus  $T^2$  in  $T^3$  defines a linear map between the skein spaces

$$e_* : K(T^2) \longrightarrow K(T^3).$$

The map  $e_*$  factorizes with the quotient map  $K(T^2) \rightarrow C(K(T^2))$ . In fact we can slide the framed links in  $e(T^2 \times [-1,1])$  from above to below getting  $e_*(L_1 \cdot L_2) = e_*(L_2 \cdot L_1)$  for every two framed links,  $L_1$  and  $L_2$ , in  $T^2 \times [-1,1]$ . As said in Remark 2.11, a standard embedding  $e : T^2 \rightarrow T^3$  corresponds to the plane generated by two columns  $(p_1, q_1, r_1), (p_2, q_2, r_2) \in \mathbb{Z}^3$  of a matrix in  $SL_3(\mathbb{Z})$ . In this correspondence  $e_*((a,b)_T) = [ap_1 + bp_2, aq_1 + bq_2, ar_1 + br_2]$  for every co-prime  $a, b \in \mathbb{Z}$ . Therefore by Lemma 2.8 we get

$$\begin{aligned} [a'p_1 + b'p_2, a'q_1 + b'q_2, a'r_1 + b'r_2] &= e_*((a', b')_T) \\ &= e_*((a, b)_T) \\ &= [ap_1 + bp_2, aq_1 + bq_2, ar_1 + br_2] \end{aligned}$$

for every two pairs  $(a,b), (a',b') \in \mathbb{Z}^2$  of co-prime integers such that  $a+a', b+b' \in 2\mathbb{Z}$ .

Let  $(p,q,r)$  be a tripe of co-prime integers. By permuting  $p,q,r$  we get either  $(p,q,r) = (1,0,0)$  or  $p,q \neq 0$ . Consider the case  $p,q \neq 0$ . Let  $d$  be the maximum common divisor of  $p$  and  $q$ , and let  $\lambda, \mu \in \mathbb{Z}$  such that  $\lambda p + \mu q = d$ . The following matrix belongs in  $SL_3(\mathbb{Z})$ :

$$M_1 := \begin{pmatrix} \frac{p}{d} & -\mu & 0 \\ \frac{q}{d} & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let  $v_1^{(1)}$  and  $v_3^{(1)}$  be the first and the third columns of  $M_1$ . We have  $(p, q, r) = dv_1^{(1)} + rv_3^{(1)}$ . Hence

$$[p, q, r] = \begin{cases} [\frac{p}{d}, \frac{q}{d}, 0] & \text{if } d \in 2\mathbb{Z} + 1, r \in 2\mathbb{Z} \\ [0, 0, 1] & \text{if } d \in 2\mathbb{Z}, r \in 2\mathbb{Z} + 1. \\ [\frac{p}{d}, \frac{q}{d}, 1] & \text{if } d, r \in 2\mathbb{Z} + 1 \end{cases}$$

The integers  $p, q, r$  can not be all even because they are co-prime, hence  $d$  and  $r$  can not be both even. Therefore we just need to study the cases where  $r \in \{0, 1\}$ .

If  $r = 0$  we consider the trivial embedding of  $T^2$  in  $T^3$ . The corresponding matrix of  $SL_3(\mathbb{Z})$  is the identity. We have  $(p/d, q/d, 0) = p/d(1, 0, 0) + q/d(0, 1, 0)$ , hence

$$[p, q, 0] = [\frac{p}{d}, \frac{q}{d}, 0] = \begin{cases} [1, 0, 0] & \text{if } \frac{p}{d} \in 2\mathbb{Z} + 1, \frac{q}{d} \in 2\mathbb{Z} \\ [0, 1, 0] & \text{if } \frac{p}{d} \in 2\mathbb{Z}, \frac{q}{d} \in 2\mathbb{Z} + 1. \\ [1, 1, 0] & \text{if } \frac{p}{d}, \frac{q}{d} \in 2\mathbb{Z} + 1 \end{cases}$$

If  $r = 1$  we take the following matrix of  $SL_3(\mathbb{Z})$ :

$$M_2 := \begin{pmatrix} 0 & 0 & 1 \\ q & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let  $v_1^{(2)}$  and  $v_3^{(2)}$  be the first and the third columns of  $M_2$ . We have  $(p, q, 1) = pv_1^{(2)} + v_3^{(2)}$ , hence

$$[p, q, 1] = \begin{cases} [1, q, 1] & \text{if } p \in 2\mathbb{Z} + 1 \\ [0, q, 1] & \text{if } p \in 2\mathbb{Z} \end{cases}.$$

By permuting  $p, q, r$  we reduce the case  $(p, q, r) = (0, q, 1)$  to the case  $p, q \neq 0, r = 0$  that we studied before.

It remains to consider the case  $p = r = 1$ . We consider the following matrix of  $SL_3(\mathbb{Z})$ :

$$M_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Let  $v_1^{(3)}$  and  $v_2^{(3)}$  be the first and the second columns of  $M_3$ . We have  $(1, q, 1) = v_1^{(3)} + qv_2^{(3)}$ . Hence

$$[1, q, 1] = \begin{cases} [1, 0, 1] & \text{if } q \in 2\mathbb{Z} \\ [1, 1, 1] & \text{if } q \in 2\mathbb{Z} + 1 \end{cases}.$$

□

**Lemma 2.13.** *The intersection of any two different standard embedded 2-tori in  $T^3$  contains a  $(p, q, r)$ -type curve.*

*Proof.* Let  $T_1$  and  $T_2$  be two standard embedded tori in the 3-torus, and let  $\pi_1$  and  $\pi_2$  be two planes in  $\mathbb{R}^3$  whose projection under the standard lattice is respectively  $T_1$  and  $T_2$ . The intersection  $T_1 \cap T_2$  contains the projection of  $\pi_1 \cap \pi_2$ . We just need to prove that in  $\pi_1 \cap \pi_2$  there is a point  $(p, q, r) \neq (0, 0, 0)$  with integer coordinates  $p, q, r \in \mathbb{Z}$ . Every plane defining a standard embedded torus is generated by two vectors with integer coordinates, and hence it is described by an equation  $ax + by + cz = 0$  with integer coefficients  $a, b, c \in \mathbb{Z}$ . Applying a linear map described by a matrix of  $SL_3(\mathbb{Z})$  we can suppose that  $\pi_1$  is the trivial plane  $\{z = 0\}$ . Let  $a, b, c \in \mathbb{Z}$  such that  $\pi_2 = \{ax + by + cz = 0\}$ . The vector  $(-b, a, 0)$  is non null and lies on  $\pi_1 \cap \pi_2$ .  $\square$

**2.5. Diagrams.** Framed links in  $T^3$  can be represented by diagrams in the 2-torus  $T^2$ . These diagrams are like the usual link diagrams but with further oriented signs on the edges (see Fig. 1-(left)). Fix a standard embedded 2-torus  $T$  in  $T^3$ . After a cut along a parallel copy  $T'$  of  $T$ , the 3-torus becomes diffeomorphic to  $T \times [-1, 1]$  and framed links in  $T^3$  correspond to framed tangles of  $T \times [-1, 1]$ . These diagrams are generic projections on  $T$  of the framed tangles in  $T \times [-1, 1]$  via the natural projection  $(x, t) \mapsto x$ . The further signs on the diagrams represent the intersection of the framed links with the boundary  $T \times \{-1, 1\}$ , in other words they represent the passages of the links along the  $(p, q, r)$ -type curve that in the Euclidean metric is orthogonal to  $T$  (see Fig. 1-(right)). If  $T$  is the trivial torus  $S^1 \times S^1 \times \{x\}$ , the further signs represent the passages through the third  $S^1$ -factor. We use the proper notion of blackboard framing.

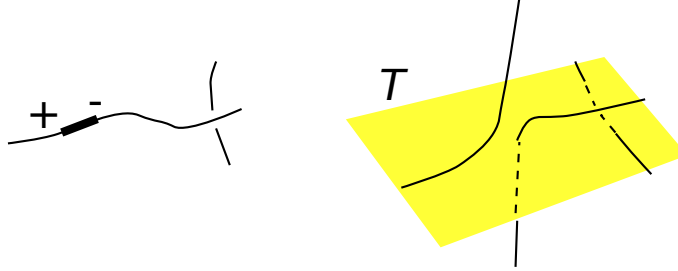


FIGURE 1. Diagrams of framed links in  $T^3$ . The yellow plane is a part of the standard embedded torus  $T \subset T^3$  where the links project. If we look at the framed links in  $T^3$  as framed tangles in  $T \times [-1, 1]$ , the two strands that get out vertically from the yellow plane end in the boundary points  $(x, 1)$  and  $(x, -1)$  for some  $x \in T$ .

**2.6. Generators for the 3-torus.** The following is the main theorem proved in this paper. We use all the previous lemmas to get a set of 9 generators of  $K(T^3)$ .



**Theorem 2.14.** *The skein space  $K(T^3)$  of the 3-torus  $T^3$  is generated by the empty set  $\emptyset$ ,  $[1, 0, 0]$ ,  $[0, 1, 0]$ ,  $[0, 0, 1]$ ,  $[1, 1, 0]$ ,  $[1, 0, 1]$ ,  $[0, 1, 1]$ ,  $[1, 1, 1]$  and a skein  $\alpha$  that is equal to the framed link consisting of two parallel copies of any  $(p, q, r)$ -type curve.*

*Proof.* Let  $T$  be the trivial embedded 2-torus: the one containing the  $(p, q, r)$ -type curves with  $r = 0$ . Use  $T$  to project the framed links and make diagrams. By using the first skein relation on these diagrams we can see that  $K(T^3)$  is generated by the framed links described by diagrams on  $T$  without crossings. These diagrams are union of simple closed curves on  $T$  equipped with some signs as the one with  $+$  and  $-$  in Fig. 1. These simple closed curves are either parallel to a  $(p, q)$ -curve or homotopically trivial. The framed links described by these diagrams lie in the standard embedded tori that are the projection (under the standard lattice) of the planes generated by  $(0, 0, 1)$  and  $(p, q, 0)$  for some  $p$  and  $q$ . Therefore  $K(T^3)$  is generated by the images of  $K(T^2)$  under the linear maps induced by the standard embeddings of  $T^2$  in  $T^3$ .

As said in the proof of Lemma 2.12, the linear map  $e_*$  induced by any standard embedding  $e : T^2 \rightarrow T^3$  factorizes with the quotient map  $K(T^2) \rightarrow C(K(T^2))$ . Lemma 2.8 applied on the standard embedding  $e$  shows that the image  $e_*(K(T^2))$  is generated by  $\emptyset$ , three  $(p, q, r)$ -type curves lying on  $e(T^2)$ , and the skein  $\alpha_e$  that is equal to the framed link consisting of two parallel copies of any  $(p, q, r)$ -type curve lying on  $e(T^2)$ .

Let  $e_1, e_2 : T^2 \rightarrow T^3$  be two standard embeddings. By Lemma 2.13  $e_1(T^2) \cap e_2(T^2)$  contains a  $(p, q, r)$ -type curve  $\gamma$ , hence  $\alpha_{e_1}$  and  $\alpha_{e_2}$  coincide with the framed link that is two parallel copies of  $\gamma$ . Therefore the skein element  $\alpha_e$  does not depend on the embedding  $e$ .

We conclude by using Lemma 2.12 that says that the skein of any  $(p, q, r)$ -type curve is equal to the one of a standard representative of a non null element of the first homology group  $H_1(T^3; \mathbb{Z}_2)$  with coefficient in  $\mathbb{Z}_2$ , namely a  $(p, q, r)$ -type curve with  $p, q, r \in \{0, 1\}$ .  $\square$

**Remark 2.15.** Theorem 2.14, Lemma 2.8 and Lemma 2.12 work for every base pair  $(R, A)$  such that  $A^n - 1$  is an invertible element of  $R$  for any  $n > 0$ . In particular they work for  $(\mathbb{C}, A)$ , where  $A^n \neq 1$  for any  $n > 0$ .

**2.7. Linear independence.** Here we talk about the linear independence of generators of  $K(T^2)$  we have shown. The following proposition shows a decomposition in direct sum of  $K(T^3)$ .

**Proposition 2.16.** *The skein space  $K(T^3)$  is the direct sum of 8 sub-spaces*

$$K(T^3) = V_0 \oplus V_1 \oplus \dots \oplus V_7$$

*such that:*

- (1)  $V_0$  is generated by the empty set  $\emptyset$  and the skein  $\alpha$  (see Theorem 2.14);
- (2) every  $(p, q, r)$ -type curve generates a  $V_j$  with  $j > 0$  and every  $V_j$  with  $j > 0$  is generated by one such curve.

*Proof.* The skein relations relates framed links holding in the same  $\mathbb{Z}_2$ -homology class. Hence for every oriented 3-manifold  $M$  we have a decomposition in direct sum

$$KM(M; R, A) = \bigoplus_{h \in H_1(M; \mathbb{Z}_2)} V_h,$$

where  $V_h$  is generated by the framed links whose  $\mathbb{Z}_2$ -homology class is  $h$ . The statement follow by this observation and the fact that if  $[p, q, r]$  and  $[p', q', r']$  represent the same  $\mathbb{Z}_2$ -homology class, then  $[p, q, r] = [p', q', r'] \in K(T^3)$ .  $\square$

**Remark 2.17.** Given a triple of integers  $(x, y, z) \neq (0, 0, 0)$  such that  $x, y, z \in \{0, 1\}$ , we can easily find an orientation preserving diffeomorphism of the 3-torus  $T^3$  sending  $[x, y, z]$  to  $[1, 0, 0]$ . Hence if the skein of one such curve  $[x, y, z]$  is null then also all the others skein elements of such curves are null. Therefore by Proposition 2.16 the possibly dimensions of the skein space  $K(T^3)$  are 0, 1, 2, 7, 8 and 9.

**Question 2.18.** Is 9 the dimension of the skein vector space  $K(T^3)$  of the 3-torus?

## REFERENCES

- [B1] D. Bullock, *The  $(2, \infty)$ -skein module of the complement of a  $(2, 2p + 1)$ -torus knot*, J. Knot Theory Ramifications 4 (4) (1995), 619–632.
- [B2] D. Bullock, *Rings of  $Sl_2(\mathbb{C})$ -characters and the Kauffman bracket skein module*, Proc. Amer. Math. Soc., 125 (6) (1997), 1835–1839.
- [BHMV] C. Blanchet, N. Habegger, G. Masbaum and P. Vogel, *Topological quantum field theories derived from the Kauffman bracket*, Topology 34 (4) (1995), 883–927.
- [C] F. Costantino, *Colored Jones invariants of links in  $\#_k S^2 \times S^1$  and the Volume Conjecture*, J. Lond. Math. Soc. 76 (2007), 1–15.
- [FG] C. Frohman and R. Gelca, *Skein modules and the non commutative torus*, Trans. Amer. Math. Soc. 352 (2000), 4877–4888.
- [HP1] J. Hoste and J. Przytycki, *A survey of skein module of 3-manifolds*, Knots 90 De Gruyter-Berlin (1992), 362–379.
- [HP2] J. Hoste and J. Przytycki, *The  $(2, \infty)$  skein module of lens spaces; a generalization of the Jones polynomial*, J. Knot Theory Ramifications 2 (1993), 321–333.
- [HP3] J. Hoste and J. Przytycki, *The skein module of genus 1 Whitehead type manifolds*, J. Knot Theory Ramifications 4 (3) (1995), 411–427.
- [HP4] J. Hoste and J. Przytycki, *The Kauffman bracket skein module of  $S^1 \times S^2$* , Mathematische Zeitschrift 220 (1995), 65–73.
- [K] L.H. Kauffman, *New invariants in the theory of knots*, Amer. Math. Monthly 95 (3) (1988) 195–242.
- [KL] L.H. Kauffman and S.L. Lins, “Temperley-Lieb recoupling theory and invariants of 3-manifolds”, Princeton University Press (1994).
- [L1] W.B.R. Lickorish, *Three-manifolds and the Temperley-Lieb algebra*, Math. Ann. 290 (1990), 657–670.
- [L2] W.B.R. Lickorish, *Calculation with the Temperley-Lieb algebra*, Comm. Math. Helv. 67 (1992), 571–591.
- [L3] W.B.R. Lickorish, *Skein and handlebodies*, Pac. J. Math. 159 (1993), 337–349.
- [L4] W.B.R. Lickorish, *The skein method for 3-manifolds invariants*, J. Knot Theory Ramifications 2 (1993), 171–194.
- [Ma] J. Marché, *The skein module of torus knots complements*, arXiv:1001.2436 (2010).

- [Mr1] M. Mroczkowski, *Kauffman bracket skein module of the connected sum of two projective spaces*, J. Knot Theory Ramifications 20 (5) (2011), 651–675.
- [Mr2] M. Mroczkowski, *Kauffman bracket skein module of a family of prism manifolds*, J. Knot Theory Ramifications 20 (1) (2011), 159–170.
- [MrD] M. Mroczkowski and M.K. Dabkowski, *KBSM of the product of a disk with two holes and  $S^1$* , Topology and its Applications 156 (2009), 1831–1849.
- [P1] J.H. Przytycki, *Fundamentals of Kauffman Bracket Skein Modules*, [arXiv:math.GT/9809113](#) (1998).
- [P2] J.H. Przytycki, *Kauffman bracket skein module of a connected sum of 3-manifolds*, Manuscripta Math. 101 (2000), 199–207.
- [P3] J.H. Przytycki, “Knots”, [arXiv:math/0602264](#) (2006).
- [TL] Thang T.Q. Le, *The colored Jones polynomial and the A-polynomial of 2-bridge knots*, presentation at the Mini-Conference in Logic and Topology, GWU (2004).
- [Tu] V.G. Turaev, *The Conway and Kauffman modules of a solid Torus*, (translation) J. Soviet. Math. 51 (1) (1990), 2799–2805.

DIPARTIMENTO DI MATEMATICA, LARGO PONTECORVO 5, 56127 PISA, ITALY  
*E-mail address:* `carrega at mail dot dm dot unipi dot it`